A Note on P-Convexity

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Abstract. In this paper we study and characterize the structure of the linear transformations which preserve P-convex sets.

Key words. P-convex set, linear transformation.

Introduction

The main objectives of research on generalized convexity are to prove convergence of algorithms to find the global minimum of non-convex problems, and to characterize functions which possess only global minima. However, from an economic point of view, a useful generalization of convexity must fulfil two requirements. First, and foremost, it must admit a rich class of economic models. Second, it must be tractable for analysis, i.e., it must have some useful mathematical structure.

In recent papers First, Hackman and Passy introduced and studied the new concept of P-convex set as a natural extension of classical convexity (see [1], [2], [3] and [4]). This concept is not only potentially rich from the applications point of view (it admits a large class of economical models), but it has consistent mathematical structure too. For instance, for P-convex sets the following separation property holds: a point not in a closed P-convex set can be separated by a quadrant, which is an intersection of closed half-spaces generated by orthogonal hyperplanes. As another example, the gradient of a differentiable P-concave function (if non-zero) generates a supporting quadrant to boundary points of the level sets. Nevertheless, unlike convex sets, P-convex sets turn out not to be invariant with respect to the general linear transformations of the Euclidean space they are immersed in.

In this paper we study and characterize the structure of the affine nonsingular maps with the property of preserving P-convex sets, that is the image under such a transformation of a P-convex set is still P-convex with respect to the previous decomposition of the space. As a consequence, we provide a large number of linear transformations that preserve the class of P-convex functions.

Notation

Throughout this paper we shall denote by X, Y, X_i finite-dimensional Euclidean spaces, by n the dimension of X, by n_i the dimension of X_i. A path in X is a not necessarily continuous map Π: $[0, 1] \to X$. If x, y are elements of X, then $[x, y]$ denotes the segment joining x with y. If x, y belong to X, then $X = \Pi_{i=1}^{m} X_i$, $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m)$ denotes the rectangle with vertices x and y and defined by

$$R_{x,y} = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_m, y_m] = \prod_{i=1}^{m} [x_i, y_i]$$

(here $[x_i, y_i]$ will denote the segment in X_i with end-points x_i, y_i.) π_i will denote the i-th projection of X onto X_i. We define in X the quadrant $Q(a)$ generated by a of X as follows

$$Q(a) = \{x \in X: a_i x_i \geq 0,\ i = 1, 2, \ldots, m\},$$

where $a_i x_i$ denotes the usual inner product in X_i.

If C is a subset of X, the symbols C°, $\overline{C}$ and C^{c} denote, respectively, the interior, the closure and the complement of C in X.

We shall denote by $M(n, R)$ the set of the square matrices of dimension n, and by $GL(n, R)$ the subset of $M(n, R)$ of the non-singular matrices.

1. Definitions and Preliminary Results

Let $X = \Pi_{i=1}^{m} X_i$ and fix this decomposition of X as a direct product of Euclidean spaces once for all; a point x which belongs to X is singled out by the m-tuple $(x_1, x_2, \ldots, x_m)$ where $x_i \in X_i$. Recall that a set $C \subseteq X$ is said to be convex if for every $x, y \in C$ we have that $\lambda x + (1 - \lambda) y \in C$, $\forall \lambda \in [0, 1]$.

It is possible to generalize (see [1]) the concept of convexity in the following way.

DEFINITION 1.1. Let C be a subset of X. C is said to be P-convex if for every $x, y \in C$, for every i $(i = 1, 2, \ldots, m)$ and for every $\lambda \in [0, 1]$, there exists $\lambda_j \in [0, 1]$, $j \neq i$ such that the point with coordinates $\lambda_j x_j + (1 - \lambda_j) y_j$ if $j \neq i$ and $\lambda x_i + (1 - \lambda) y_i$ if $j = i$ belongs to C. Notice that the definition of P-convex set is strictly related to the decomposition of the space X into the direct product $\Pi_{i=1}^{m} X_i$.

The concept of P-convexity can be as well formulated as follows: for every couple of points $x, y \in C$ and for every index i $(i = 1, 2, \ldots, m)$, there exists a path Π from x to y such that its image is contained in $R_{x,y} \cap C$ and such that the projection $\pi_i(\Pi)$ is exactly the segment $[x_i, y_i]$.

In the special case where C is a closed subset of X, then Hackman and Passy stated the following

PROPOSITION 1.2. *Let C be a P-convex set; then, for every point $x \in \partial C$ there exists a support quadrant $Q(a) + x$, that is a quadrant such that*

$$(Q(a) + x) \cap C^0 = \emptyset .$$

In particular, a P-convex closed set is the intersection of the sets which are complements of quadrants containing it. The condition expressed by proposition 1.2 cannot be reversed.

EXAMPLE. Let $X = R \times R$, $a = (1, 1)$, $b = (-1, 1)$. The closed set $C = [0, a] \cup [0, b]$ is not P-convex, but there exists for every point at least one support quadrant.

Let now $f : X \to \boldsymbol{R}$.

DEFINITION 1.3. f is said to be P-convex (P-concave) on X if the lower level sets (upper level sets) are P-convex.

2. Invariance under Linear Transformations

Let $X = \Pi_{i=1}^m X_i$, $\dim X = n$. Recall that any transformation of the group $M(n, R)$ preserves convex sets of X (the proof of that the property follows directly from the definition of convex set). Indeed, if x, y are points in the convex set C, we have $[x, y] \subset C$. Let $T \in M(n, R)$; since T is linear, then $T[x, y] = [Tx, Ty] \subseteq TC$, which is the thesis. Let us consider now the P-convex set $C \subset X$. Let $T \in GL(n)$. We put the following questions:
(i) is the set

$$T^{-1}C = \{y \in X : y = T^{-1}x, x \in C\}$$

P-convex?
(ii) which kinds of transformations $T \in GL(n, R)$ do preserve the P-convexity of a set?

The answer to the first question is negative. Indeed it is enough to consider in $X = R \times R$ the P-convex set $C = Q^c$, with $Q = Q(1, 1)$. Assume that T is the matrix

$$\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix},$$

that is a rotation characterized by an angle θ, with $\theta \neq k\pi/2, k \in Z$; it can be immediately verified that the set TC is not P-convex.

We shall see in the sequel that the linear transformations persevering P-convexity are comparatively few. We shall prove the following.

THEOREM 2.1. *Let $T \in GL(n, R)$. Therefore T preserves the P-convex sets of X if and only if there exists a one-to-one transformation α of the set $\{1, 2, \ldots, m\}$ such that*

$$T(X_i) = X_{\alpha(i)}$$

for every $i = 1, 2, \ldots, m$. *In particular* $n_i = n_{\alpha(i)}$

To proceed we need some technical lemmata.

LEMMA 2.2. *Let* $T \in GL(n, R)$, $T = T_1 \oplus T_2 \oplus \cdots \oplus T_m$, $T_i \in O(n_i)$. *Then* T *preserves the P-convex subsets of* X.

Proof. It suffices to show that for all x, $y \in X$ we have

$$TR_{x,y} = R_{Tx,Ty} .$$

Set $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$. We have

$$Tx = (T_1x_1, \ldots, T_mx_m), \quad Ty = (T_1y_1, \ldots, T_my_m), \quad \pi_i(Tx) = T_ix_i ,$$
$$\pi_i(Ty) = T_iy_i .$$

Since T_i and π_i commute, then

$$\begin{aligned} R_{Tx,Ty} &= [\pi_1(Tx), \pi_1(Ty)] \times \cdots \times [\pi_m(Tx), \pi_m(Ty)] \\ &= T([\pi_1x, \pi_1y] \times \cdots \times [\pi_mx, \pi_my]) \\ &= TR_{x,y} . \end{aligned}$$

which gives the thesis. □

LEMMA 2.3. *Let* $T \in GL(n, R)$ *and assume that there exists a transformation* α *of the set* $\{1, 2, \ldots, m\}$ *such that*

$$TX_i = X_{\alpha(i)} , \quad i = 1, 2, \ldots, m .$$

Then T *preserves the P-convex subsets.*

Proof. It is a trivial generalization of the proof above. □

From now on we denote by $\{e^i_k\}_{k=1}^{n_i}$ the canonical basis of the space X_i, and by $\pi^i_k(t)$ the hyperplane of X defined by the equation $x^i_k = t$, $t \in R$.

LEMMA 2.4. *If* $a, b \in \pi^i_k(t)$, *then the rectangle* $R_{a,b}$ *is contained in* $\pi^i_k(t)$.

Proof. $R_{a,b} = \Pi_{i=1}^m [a_i, b_i]$. Since $a^i_k = b^i_k = t$, we get

$$[a_i, b_i] = \bigcup_{\lambda \in [0,1]} \{\lambda a^i_1 + (1-\lambda)b^i_1, \ldots, t, \ldots, \lambda a^i_m + (1-\lambda)b^i_m\} ,$$

that is to say $R_{a,b} \subset \pi^i_k(t)$. □

Identify in a natural way a point a in X_i with the point of X, $x = (0, 0, \ldots, 0, a, 0, \ldots, 0)$ (a is in the i-th position), and denote by X^T_i the set of points with i-th coordinate equal to zero.

REMARK 2.5. Let $a \in X_i, b \in X_i^{\perp}$. Then the set $C = [0, a] \cup [0, b]$ is a P-convex subset of X.

PROPOSITION 2.6. *Let $T \in GL(n, R)$ be a linear transformation preserving the P-convex sets of X. Therefore one and only one of the following properties holds: $TX_i = X_i$ or $TX_i \subseteq X_i^{\perp}$, for all $i = 1, 2, \ldots, m$.*

Proof. Assume that $i = 1$ and $TX_1 \neq X_1$, and that there exist indices i, k and $t \neq 0$ such that

$$TX_1 \cup \pi_k^i(t) \neq \emptyset\,, \quad TX_1^{\perp} \cap \pi_k^i(t) \neq \emptyset\,;$$

let $a \neq 0$, $a \in TX_1 \cap \pi_k^i(t)$, $b \neq 0$, $b \in TX_1^{\perp} \cap \pi_i^k(t)$. Consider the set C defined as follows

$$C = [0, T^{-1}a] \cup [0, T^{-1}b]\,.$$

C is a P-convex subset of X since $T^{-1}a \in X_1$, $T^{-1}b \in X_1^{\perp}$ (see 2.5); on the other hand we have that

$$TC = [0, a] \cup [0, b]$$

therefore TC is not P-convex since

$$R_{a,b} \cap TC = \{a,b\}\,.$$

(see (2.4)). We can conclude that there are not indices i, k and $t \neq 0$ satisfying the conditions above. Notice that if $\pi_k^i(t) \cap TX_1 \neq \emptyset$ for some $t \neq 0$ then, by linearity, $\pi_k^i(t) \cap TX_1 \neq \emptyset$ for every $t \in R$. Define the sets N_1, N_2 in the following way:

$$N_1 = \{(i, k) : \pi_k^i(t) \cap TX_1 = \emptyset, t \neq 0\}$$
$$N_2 = \{(i, k) : \pi_k^i(t) \cap TX_1^{\perp} = \emptyset, t \neq 0\}\,.$$

N_1 and N_2 are disjoint; moreover, if $(i, k) \in N_1$, from the fact that

$$\bigcup_{t \in R} \pi_k^i(t) = X\,, \quad TX_1 \cap \Big(\bigcup_{t \neq 0} \pi_k^i(t)\Big) = \emptyset\,,$$

we get the inclusion $TX_1 \subseteq \pi_k^i(0)$. In the same way $TX_1^{\perp} \subseteq \pi_k^i(0)$ if $(i, k) \in N_2$. We can write that

$$TX_1 \subseteq \bigcap_{(i,k) \in N_1} \pi_k^i(0)$$

$$TX_1^{\perp} \subseteq \bigcap_{(i,k) \in N_2} \pi_k^i(0)$$

By an obvious dimensional computation, we can conclude that in both cases the strict equality holds.

Since, by the hypothesis, $TX_1 \neq X_1$, there exists h such that $(1, h) \in N_1$;

moreover, if $TX_1 \not\subseteq X_1^\perp$, there is l such that $(1, l) \in N_2$. Denote by C the following set

$$C = [0, T^{-1}e_h^1] \cup [0, T^{-1}e_l^1] .$$

C is P-convex in X, but the image of C under T, TC, which coincides with $[0, e_h^1] \cup [0, e_l^1]$, is not P-convex since

$$R_{e_h^1, e_l^1} \cap TC = \{e_h^1, e_l^1\} .$$

Hence $TX_1 \neq X_1 \Rightarrow TX_1 \subseteq X_1^\perp$. □

PROPOSITION 2.7. *Let $T \in GL(n, R)$ be a linear transformation of X preserving P-convexity. Then there exists a transformation α of $\{1, 2, \ldots, m\}$ such that*

$$TX_i = X_{\alpha(i)} , \quad i = 1, 2, \ldots, m$$

with $n_i = n_{\alpha(i)}$.

Proof. Assume, by simplicity, that the following relation holds: $n_1 \leqslant n_2 \leqslant \cdots n_m$. Our proof is absurd. Assume $TX_1 \neq X_i$ for every $i = 1, 2, \ldots, m$. Therefore, (see Prop. 2.6) we have that $TX_1 \subseteq X_2 \times X_3 \times \cdots \times X_m$. Since $n_1 \leqslant n_i$ for every i, there exist $(i, h) \in N_1$, $(i, l) \in N_2$ such that the set $C = [0, T^{-1}e_h^i] \cup [0, T^{-1}e_l^i]$ is P-convex whereas TC is not, contradicting the hypothesis. We can conclude that there exists $\alpha(1)$ such that $TX_1 = X_{\alpha(1)}$. In a similar way we work with $X_2, X_3, \ldots, X_m$. □

Proof of Theorem 2.1. It follows immediately from (2.3) and (2.7). □

COROLLARY 2.8. *If $f(x)$ is a P-convex function, and if T is a linear transformation satisfying the assumptions of Theorem* 2.1, *then the function $f(Tx)$ is P-convex; in particular, its true local minima are global minima.*

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